



## Brief paper

Gradient free cooperative seeking of a moving source<sup>☆</sup>Elad Michael<sup>a,\*</sup>, Chris Manzie<sup>a</sup>, Tony A. Wood<sup>b</sup>, Daniel Zelazo<sup>c</sup>, Iman Shames<sup>d</sup><sup>a</sup> The University of Melbourne, Parkville VIC 3010, Australia<sup>b</sup> SYCAMORE Lab, Ecole Polytechnique Federale de Lausanne (EPFL), Lausanne, Switzerland<sup>c</sup> TECHNION, Haifa 32000, Israel<sup>d</sup> CHICADA LAB, ANU, Canberra ACT 0200, Australia

## ARTICLE INFO

## Article history:

Received 2 January 2022

Received in revised form 19 October 2022

Accepted 29 January 2023

Available online 9 March 2023

## Keywords:

Extremum seeking

Multi-agent systems

Decentralisation

Optimisation under uncertainty

Tracking

## ABSTRACT

In this paper, we consider the optimisation of a time varying scalar field by a network of agents with no gradient information. We propose a composite control law, blending extremum seeking with formation control in order to converge to the extrema faster by minimising the gradient estimation error. By formalising the relationship between the formation and the gradient estimation error, we provide a novel analysis to prove the convergence of the network to a bounded neighbourhood of the field's time varying extrema. We assume the time-varying field satisfies the Polyak–Łojasiewicz inequality and the gradient is Lipschitz continuous at each iteration. Numerical studies and comparisons are provided to support the theoretical results.

© 2023 The Author(s). Published by Elsevier Ltd. This is an open access article under the CC BY license (<http://creativecommons.org/licenses/by/4.0/>).

## 1. Introduction

Localising the source of an unknown or uncertain scalar field has attracted significant attention in recent years. Extremum seeking can then be understood as driving the state of an agent or network of agents to the source, and maintaining a steady state in the neighbourhood of this optimal state in the unknown field. The widespread applications include internal combustion engine calibration (Killingsworth, Aceves, Flowers, Espinosa-Loza, & Krstic, 2009), locating RF leakage (Al Issa, 2012), optimising energy distribution (Ye & Hu, 2016), and mobile sensor networks (Stankovic, Johansson, & Stipanovic, 2011). The main challenge in general is the approximation of the field, or a valid descent direction, with the additional challenge in the multi-agent case of coordinating the agents to improve the estimation. In this work, we consider discrete time extremum seeking, for the more classical continuous time extremum seeking problem see Tan, Moase, Manzie, Nešić, and Mareels (2010) and the references therein.

Extremum seeking with a single agent primarily uses “dither” or other motion patterns to estimate a descent direction. In Cochran and Krstic (2009) and Zhang, Siranosian, and Krstić

(2007), extremum seeking with a single agent is investigated relying only on the measurements of the scalar field, without usage of the agent's position. Both approaches use a sinusoidal dither signal to estimate the gradient of the unknown field. Using finite difference with previous measurements, tracking and estimation error bounds for the minima of a time-varying scalar field are derived in Shames, Selvaratnam, and Manton (2019), along with extensive numerical studies using a single agent. A hybrid controller is defined in Mayhew, Sanfelice, and Teel (2007), conducting a series of line minimisations to construct the descent direction, with stability and convergence results. In Dürr, Stanković, Ebenbauer, and Johansson (2013) and Dürr, Krstić, Scheinker, and Ebenbauer (2017), the authors derive an extremum seeking controller using Lie bracket approximations of the field, however the approach is only applied to continuous time dynamics and in static fields.

Using a network of agents allows for a more robust estimate of the gradient, as the measurements are typically assumed to be simultaneous and thus unaffected by a time-varying source. In Biyik and Arcak (2008) a network is used with a single leader determining the estimated gradient, employing a zero mean dither signal, with the followers only keeping formation. The authors show that with a fast dither and slow formation keeping, the followers only track the gradient descent movement of the leader. A game theoretic approach is used in Stankovic et al. (2011) to find equilibria of each agent's individual cost functions, using local extremum seeking controllers with sinusoidal dither. Using multiple “leader” agents and only inter-agent bearing measurements, the authors in Zhao and Zelazo (2015a,

<sup>☆</sup> The material in this paper was partially presented at the 59th IEEE Conference on Decision and Control, December 14–18, 2020, Jeju Island, Republic of Korea. This paper was recommended for publication in revised form by Editor Christos G. Cassandras.

\* Corresponding author.

E-mail addresses: [eladm@student.unimelb.edu.au](mailto:eladm@student.unimelb.edu.au) (E. Michael), [manziec@unimelb.edu.au](mailto:manziec@unimelb.edu.au) (C. Manzie), [tony.wood@epfl.ch](mailto:tony.wood@epfl.ch) (T.A. Wood), [dzlazo@technion.ac.il](mailto:dzlazo@technion.ac.il) (D. Zelazo), [iman.shames@anu.edu.au](mailto:iman.shames@anu.edu.au) (I. Shames).

2015b) stabilise a formation in arbitrary dimension with leaders following reference velocities or trajectories. In addition, using only bearing measurements allows for formation scaling and rotation. Using only inter-agent distances and function measurements, the authors in Ghods and Krstic (2011) are able to deploy a swarm of agents which statistically cluster around the stationary source. In multi-agent approaches, the set of measurements from each agent can be used to compute an estimated gradient, assuming a single sensor aboard each agent (Khong, Tan, Manzie, & Nešić, 2014; Ogren, Fiorelli, & Leonard, 2004; Skobeleva, Fidan, Ugrinovskii, & Petersen, 2018; Vandermeulen, Guay, & McLellan, 2017; Vweza, Chong, & Lee, 2015). Many of these publications use some form of the simplex gradient (Regis, 2015), as do we in this paper. The controller design derived in Khong et al. (2014) uses a centralised extremum seeking controller, with access to all of the agents' measurements, which provides reference velocities to each of the agents. Convergence guarantees are provided for a variety of formation and extremum seeking methods satisfying their assumptions. A centralised controller is implemented in Ogren et al. (2004) to track the estimated gradient using least squares estimation and refined by Kalman filtering. The agents are tasked with formation keeping around a virtual leader, which climbs the gradient of the unknown field. However, the problem formulation only considers finite manoeuvres, and the formation may move extremely slowly. A centralised controller and agents with unicycle dynamics are examined in Frihauf, Liu, and Krstic (2014), in which they provide an algorithm which does not measure the agents' positions and provably converges to the source. For networks of 3 agents in 2 dimensions a distributed control law with exponential convergence guarantees is investigated in Skobeleva et al. (2018). The agents in Vandermeulen et al. (2017) use a dynamic consensus algorithm to coordinate the gradient estimation, combined with a zero mean dither to construct a local gradient estimation. Finally, in a series of papers (Brinón-Arranz & Schenato, 2013; Brinón-Arranz, Schenato, & Seuret, 2015; Brinón-Arranz, Seuret, & Canudas-de Wit, 2011; Moore & Canudas-de Wit, 2010), a group of unicycle agents performing distributed extremum seeking in circular formations is examined. The agents stabilise their formation and gradient estimate using a consensus algorithm, and performs well even with lossy communication and time-varying communication networks. The algorithm described in Brinón-Arranz et al. (2015) is implemented in Section 5 to compare to the results derived in this paper.

Recently, extremum seeking for sources with dynamics has received some attention. In Section 1.2 of Ariyur and Krstic (2003), an extremum seeking algorithm using the internal model principle is derived, but requires extensive internal knowledge of the plant's dynamics. In Poveda and Krstić (2021), the authors derive similar tracking results to those provided here, albeit using continuous time dynamics and with the assumption of strong convexity using a non-smooth extremum seeking controller. Several recent works address time-varying extremum seeking in continuous time using a periodic dither algorithm (Grushkovskaya, Dürr, Ebenbauer, & Zuyev, 2017; Hazeleger, Haring, & van de Wouw, 2020; Moshksar, Dougherty, & Guay, 2015), however these works assume that the source/plant variation is significantly slower than the dither speed to allow for gradient estimation.

**Contributions.** This paper provides a novel analysis of multiagent extremum seeking focused on a time-varying source without using a centralised coordinator or dither motion, with discrete dynamics. This differs from the majority of the literature, which assumes a static or slowly drifting scalar field. In this work,

- we allow the scalar field to be time-varying with no constraints on periodicity or time-scale separation;

- we incorporate formation control into extremum seeking using a novel condition on the formation potential, formalising the relationship between the gradient estimation and the formation;
- we show that the agents converge to a bounded neighbourhood of the time-varying extrema of the field;
- we present two elliptical error bounds on the gradient approximation of a function with a Lipschitz continuous gradient.
- we provide an open-source implementation of the approach to allow for further research and validation of our results.

Finally, at each iteration, we only assume that the time varying field is represented by a function which has Lipschitz continuous gradient (bounded second derivative), and satisfies the Polyak–Łojasiewicz (PL) inequality. The PL inequality assumption is also weaker than many which are used to provide the linear convergence of gradient descent algorithms, such as convexity or quadratic growth (Karimi, Nutini, & Schmidt, 2016). For further information on invex functions, see Ben-Israel and Mond (1986). The authors' previous investigation into this problem (Michael, Zelazo, Wood, Manzie, & Shames, 2020) included a more complicated control law than is presented here, with results restricted to 2 dimensions. In this analysis, we simplify the control law, derive stronger convergence guarantees, and broaden the method to arbitrary dimension.

The paper is organised as follows. Section 2 is devoted to basic assumptions on the time-varying field and agent dynamics. Section 3 discusses the distributed control law and its performance for extremum seeking and formation keeping. Section 4 provides an example of cooperative gradient estimation, an improvement and generalisation of the results from Michael et al. (2020). We provide numerical simulations in Section 5, and conclude in Section 6.

## 2. Problem formulation

Consider a network of  $n$  agents where  $x_k^{(i)} \in \mathbb{R}^d$  denotes the position of the  $i$ th agent for  $i \in \{1, \dots, n\}$  at iteration  $k$ . We use bold variables throughout the paper to describe the stacked vector for all agents, i.e.  $\mathbf{x}_k$  to denote the vector of all agents' states stacked vertically. Let  $\mathcal{G} = (\mathcal{V}, \mathcal{E})$  be the underlying graph of the network with the vertex set  $\mathcal{V} = \{1, \dots, n\}$  representing the agents and the edge set  $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$  representing the communication topology. For each agent  $i$ , we define a set of neighbours  $\mathcal{N}^{(i)} := \{j \mid (j, i) \in \mathcal{E}\}$  from which agent  $i$  receives information at each iteration step.

**Assumption 1.** Assume that the agent communication graph  $\mathcal{G} = (\mathcal{V}, \mathcal{E})$  is connected and time invariant.

The agents are modelled as single integrators:

$$\dot{x}_{k+1}^{(i)} = x_k^{(i)} + \alpha p_k^{(i)}, \quad (1)$$

where  $\alpha$  is a constant.

**Remark 1.** We assume the single integrator dynamics (1) to focus on the time-varying field and highlight the extremum seeking algorithm used. However, the proposed approach may provide waypoints for a lower level controller, which navigates on a faster timescale until the waypoint is reached and the next measurement collected. The extension from single integrator dynamics to more complicated dynamics including velocity saturated models and nonholonomic models is discussed in Zhao and Sun (2017).

At each iteration  $k$ , the time-varying field is represented by the function  $f_k : \mathbb{R}^d \rightarrow \mathbb{R}$  with the non-empty minimiser set  $\mathcal{X}_{f_k}^* := \operatorname{argmin}_{x \in \mathbb{R}^d} f_k(x)$ . The agents can only measure the function value at their location at each iteration, i.e. the value  $f_k(x_k^{(i)})$ . For any dimension  $m \in \mathbb{Z}^+$  we define the distance between a point  $x \in \mathbb{R}^m$  and a set  $\mathcal{S} \subseteq \mathbb{R}^m$  as  $d(x, \mathcal{S}) = \inf_{y \in \mathcal{S}} \|y - x\|$ , where  $\|\cdot\|$  is the Euclidean norm. Additionally, for a function  $h : \mathcal{D} \rightarrow \mathbb{R}$  we will use  $h^*$  to represent the minimum value of that function over its domain.

**Assumption 2** (*Differentiability and Lipschitz Gradient*). For all  $k \geq 0$ , the functions  $f_k : \mathbb{R}^d \rightarrow \mathbb{R}$  are at least once continuously differentiable. The gradients are  $L_f$ -Lipschitz continuous, i.e. there exists a positive scalar  $L_f$  such that, for all  $k \geq 0$ ,  $x \in \mathbb{R}^d$ ,  $y \in \mathbb{R}^d$ ,  $\|\nabla f_k(x) - \nabla f_k(y)\| \leq L_f \|x - y\|$ , or equivalently  $f_k(y) \leq f_k(x) + \nabla f_k(x)^T(y - x) + \frac{L_f}{2} \|y - x\|^2$ .

**Assumption 3** (*Polyak–Lojasiewicz (PL) Condition*). For all  $k \geq 0$ , there exists a positive scalar  $\mu_f$  such that  $\frac{1}{2} \|\nabla f_k(x)\|^2 \geq \mu_f (f_k(x) - f_k^*)$ .

The assumption that a function has an  $L$ -Lipschitz continuous gradient is equivalent to assuming the second derivative has bounded norm, if it is twice differentiable. The PL condition requires that the gradient grows faster than a quadratic as we move away from the optimal function value. The PL condition does not require the minima to be unique, although it does guarantee that every stationary point is a global minimum (Karimi et al., 2016). In addition to Assumptions 2–3 on each  $f_k$ , we quantify the “speed” with which the field may vary next.

**Assumption 4** (*Bounded Drift in Time*). There exist positive scalars  $\eta_0$  and  $\eta^*$  such that  $|f_{k+1}(x) - f_k(x)| \leq \eta_0$  for all  $x \in \mathbb{R}^d$  and  $|f_k^* - f_{k+1}^*| \leq \eta^*$ .

The problem of interest is given below.

**Problem 5.** For a network of  $n$  agents with dynamics (1) and communication topology satisfying Assumption 1, let  $\{f_k\}$  be a sequence of functions with a corresponding sequence of minimiser sets  $\{\mathcal{X}_{f_k}^*\}$  satisfying Assumptions 2–4. Given the measurements  $\mathcal{Y}_k^{(i)} = \{f_k(x_k^{(j)}) \mid j \in \mathcal{N}^{(i)} \cup \{i\}\}$ , find  $\alpha$ ,  $p_k^{(i)}$  and a constant  $M$  for all agents  $i \in \mathcal{V}$  and for all  $k \geq 0$  such that  $\lim_{k \rightarrow \infty} d(x_k^{(i)}, \mathcal{X}_{f_k}^*) \leq M$ .

In Section 3, we will incorporate formation control into the extremum seeking algorithm. To this end, we use a formation potential function  $\phi(\mathbf{x}_k) : \mathbb{R}^{nd} \rightarrow \mathbb{R}^+$  which takes the full state vector of all agents and returns a scalar which is minimised when the agents are in formation. Let the minimum be denoted by  $\phi^* := \min_{\mathbf{x} \in \mathbb{R}^{nd}} \phi(\mathbf{x})$ .

**Definition 6.** We define  $\phi : \mathbb{R}^{nd} \rightarrow \mathbb{R}^+$  to be the *formation potential function* for the network, with minimisers  $\mathcal{X}_\phi^*$ . Moreover, the function  $\phi(\mathbf{x}_k)$  (1) is continuously differentiable on  $\mathbb{R}^{nd}$  with gradient which is Lipschitz continuous with constant  $L_\phi$ ; (2) satisfies the PL inequality (Assumption 3), with constant  $\mu_\phi \geq \mu_f$ ; (3) has gradient component  $\nabla_{x_k^{(i)}} \phi(\mathbf{x}_k)$  which is computable using only the state of agent  $i$  and neighbours  $j \in \mathcal{N}_i$ .

In the definition of the formation potential functions, the first two conditions ensure that  $\phi(\mathbf{x}_k)$  shares the minimal properties that make  $f_k$  amenable to analysis. The third property ensures that the local information each agent has is sufficient for computation of the descent direction. Navigation potential functions such as in De Gennaro and Jadbabaie (2006), Dimarogonas, Loizou, Kyriakopoulos, and Zavlanos (2006), Do (2006), Olfati-Saber and Murray (2002) and Tanner and Kumar (2005) satisfy these assumptions. We provide an example of a formation potential function that satisfies these assumptions in Section 5.

### 3. Cooperative gradient descent

In this section we show that a network of agents cooperating can reach a bounded neighbourhood of the minimiser set. In this section, for simplicity, we assume each agent uses an  $\varepsilon$ -gradient oracle at each iteration to construct a step direction.

**Definition 7** ( $\varepsilon$ -gradient Oracle). Given the function  $f_k : \mathbb{R}^d \rightarrow \mathbb{R}$  and the state of the agents  $\mathbf{x}_k \in \mathbb{R}^d$ , the oracle returns  $O(f_k, \mathbf{x}_k, \mathcal{N}^{(i)}) = \nabla f_k(x_k^{(i)}) + \varepsilon_k$ .

In order to motivate the incorporation of formation control, consider the case where  $p_k^{(i)} = -O(f_k, \mathbf{x}_k, \mathcal{N}^{(i)})$ :

$$x_{k+1}^{(i)} = x_k^{(i)} - \alpha (\nabla f_k(x_k^{(i)}) + \varepsilon_k). \quad (2)$$

We have the following lemma (proof in Appendix A) on the convergence properties of the system.

**Lemma 8.** For a sequence of functions  $\{f_k\}$  with minimiser sets  $\{\mathcal{X}_{f_k}^*\}$  satisfying Assumptions 2–4, for  $\beta \in \mathcal{K}\mathcal{L}$ ,  $\alpha \in (0, \frac{1}{L_f})$ , the system with dynamics (2) satisfies

$$\begin{aligned} \frac{1}{2} d(x_k^{(i)}, \mathcal{X}_{f_k}^*)^2 &\leq \beta (d(x_0^{(i)}, \mathcal{X}_{f_0}^*)^2, k) \\ &+ \frac{\alpha}{2\mu_f} \sum_{t=0}^k (1 - \alpha\mu_f)^{k-t} \|\varepsilon_t\|^2 + \frac{\eta_0 + \eta^*}{\alpha\mu_f^2}. \end{aligned} \quad (3)$$

**Remark 9.** Lemma 8 seems to imply that if  $\alpha$  is chosen to be  $\frac{1}{\mu_f}$ , the impact of the gradient error from steps before  $k$  is zero. To understand why, note that the Lipschitz constant  $L_f$  and PL constant  $\mu_f$  satisfy the following

$$\frac{\mu_f}{2} d(x_k^{(i)}, \mathcal{X}_{f_k}^*)^2 \leq f_k(x) - f_k^* \leq \frac{L_f}{2} d(x_k^{(i)}, \mathcal{X}_{f_k}^*)^2, \quad (4)$$

see Karimi et al. (2016) for in depth discussion regarding the PL inequality. Requiring that  $\alpha \leq \frac{1}{L_f}$  implies  $\alpha \leq \frac{1}{\mu_f}$ . Thus, if  $\alpha \approx \frac{1}{\mu_f}$ , then we must have that  $\mu_f \approx L_f$  and  $f_k$  is approximately a scaled norm as a consequence of (4). For the scaled norm function, the gradient dynamics (2) would take the agent directly to the minimiser, except for the error term from the most recent gradient estimate in (3) and the drift error term  $\frac{\eta_0 + \eta^*}{\mu_f}$ .

From Lemma 8, the system with dynamics (2) converges to a neighbourhood dependent on the magnitude of the gradient error terms  $\|\varepsilon_k\|^2$  and a constant term due to drift. This result is similar to the  $(\beta, \gamma)$ -tracking property defined in Poveda and Krstić (2021) with the power series in terms of  $\|\varepsilon_t\|^2$  as the  $\gamma$  function therein. As noted in their paper, the result from Lemma 8 resembles a semi-global practical ISS bound with respect to  $\|\varepsilon_k\|^2$  as the input. However, we can improve upon this tracking result, as the magnitude of the gradient error  $\|\varepsilon_t\|^2$  is not bounded. An idea behind this work is that in using function samples to estimate the gradient, the error in estimation is generally a function of the geometry of the samples taken. By incorporating formation control into the dynamics, we are able to bound the error terms  $\|\varepsilon_k\|^2$ . We show a specific example of this in Section 4, but make minimal assumptions in this section on the specifics of how to construct a gradient estimate from sample points.

To characterise the entire network’s behaviour, we define the time-varying function  $F_k : \mathbb{R}^{nd} \rightarrow \mathbb{R}$  as  $F_k(\mathbf{x}_k) := \sum_{i \in \mathcal{V}} f_k(x_k^{(i)})$ , and note that  $F_k$  satisfies Assumptions 2–3 with the same constants  $L_f, \mu_f$ . The time-varying minimiser set of  $F_k(\mathbf{x}_k)$  is  $\mathcal{X}_{F_k}^* = \left(\mathcal{X}_{f_k}^*\right)^n$ .

To incorporate formation control into the extremum seeking analysis, we make the following assumption about the selection of  $\phi(\mathbf{x}_k)$ .

**Assumption 10.** The formation potential function  $\phi : \mathbb{R}^{nd} \rightarrow \mathbb{R}^+$  as in Definition 6 satisfies  $\phi(\mathbf{x}_k) \geq \frac{c}{2} \sum_{i \in \mathcal{V}} \|\varepsilon_k^{(i)}\|^2$ , where  $\varepsilon_k^{(i)}$  is defined in Definition 7, and  $c \in \mathbb{R}^+$  chosen such that  $c > \frac{1}{\mu_f}$ .

Assumption 10 formalises the relationship between the gradient estimation error and the formation. In Section 5 we provide the example  $\phi(\mathbf{x}_k) = \phi^* + L_f \sum_{i \in \mathcal{V}} \|\mathbf{x}^{(i)} - \mathbf{x}^{(j)} - \hat{\mathbf{x}}^{(ij)}\|^2$ , where the terms  $\hat{\mathbf{x}}^{(ij)}$  define the optimal formation and the constant  $\phi^*$  ensures Assumption 10 is satisfied when the agents are in perfect formation. The constant offset does not change the dynamics, it allows  $\phi(\mathbf{x}_k)$  to bound the gradient error in the convergence analysis, see the proof of Theorem 12. This formation potential function satisfies the assumptions in Definition 6 and Assumption 10, however it requires an *a priori* selection of each agent's neighbours.

With the formation potential function defined, we define the "composite" function  $\hat{f}_k : \mathbb{R}^{nd} \rightarrow \mathbb{R}$  as

$$\hat{f}_k(\mathbf{x}_k) := F_k(\mathbf{x}_k) + \phi(\mathbf{x}_k), \quad (5)$$

with corresponding minimisers in the set  $\mathcal{X}_{\hat{f}_k}^*$ , and the new system dynamics  $\mathbf{x}_{k+1}^{(i)} := \mathbf{x}_k^{(i)} - \alpha(\nabla_{\mathbf{x}_k^{(i)}} \hat{f}_k + \varepsilon_k)$ . Each agent can compute the gradient  $\nabla_{\mathbf{x}_k^{(i)}} \phi(\mathbf{x}_k)$  with only local information, so the gradient of the composite function, being the sum of  $f_k$  and  $\phi$ , can be estimated by using the same  $\varepsilon$ -gradient oracle for  $f_k$ . Both  $F_k$  and  $\phi$  satisfy Assumption 2 with constants  $L_f, L_\phi$  respectively, and Assumption 3 with constants  $\mu_f, \mu_\phi$ . Therefore, the composite function satisfies both Assumptions 2 and 3 with constants  $L_f := L_f + L_\phi$  and  $\mu_{\hat{f}} \geq \min(\mu_f, \mu_\phi) = \mu_f$ . The following lemma with proof in Appendix B states the properties of  $\hat{f}_k$ .

**Lemma 11.** For the composite function  $\hat{f}_k$ , as defined in (5),  $\hat{f}_k^* \leq \phi^* + \frac{\min(L_f, L_\phi)}{2} d(\mathcal{X}_{F_k}^*, \mathcal{X}_\phi^*)^2$ , where  $\hat{f}_k^* := \min_{\mathbf{x} \in \mathbb{R}^{nd}} \hat{f}_k(\mathbf{x})$  and  $d(\mathcal{X}_{F_k}^*, \mathcal{X}_\phi^*) := \min\{\|\mathbf{x}_\phi^* - \mathbf{x}_{F_k}^*\| \mid \mathbf{x}_\phi^* \in \mathcal{X}_\phi^*, \mathbf{x}_{F_k}^* \in \mathcal{X}_{F_k}^*\}$ .

In the following theorem (proof in Appendix C), we show that by incorporating a formation potential function, which bounds the gradient estimation error, the agents converge to a bounded neighbourhood of the time varying minimiser set  $\mathcal{X}_{\hat{f}_k}^*$ . Furthermore, the system does not require leaders, a separate time-scale for the formation-keeping, or any centralised computation.

**Theorem 12.** For a sequence of functions  $\{\hat{f}_k\}$  as defined in (5) with minimisers  $\{\mathcal{X}_{\hat{f}_k}^*\}$ , the system with dynamics (2) satisfies  $\frac{1}{2} d(\mathbf{x}_{k+1}^{(i)}, \mathcal{X}_{\hat{f}_{k+1}}^*)^2 \leq \beta(d(\mathbf{x}_0^{(i)}, \mathcal{X}_{\hat{f}_0}^*), k) + \frac{\alpha}{c\mu} \sum_{t=0}^k (1 - \alpha\mu')^{k-t} \hat{f}_t^* + \frac{\eta_0 + \eta^*}{\alpha\mu\mu'}$ , for  $\beta \in \mathcal{KL}$ ,  $\alpha \in (0, \frac{1}{L_f}]$ , and  $\mu' = \mu_f - \frac{1}{c}$ . Moreover,

$$\limsup_{k \rightarrow \infty} \frac{1}{2} d(\mathbf{x}_{k+1}^{(i)}, \mathcal{X}_{\hat{f}_{k+1}}^*)^2 \leq \frac{\limsup_{k \rightarrow \infty} \hat{f}_k^*}{\mu'} + \frac{\eta_0 + \eta^*}{\alpha\mu\mu'}.$$

#### 4. Gradient estimation and error

In Section 3, we assume that each agent has access to an estimate of  $\nabla f(\mathbf{x}_k^{(i)}) + \varepsilon^{(i)}$ . In this section, we provide a method by which agent  $i$  can estimate  $\nabla f(\mathbf{x}_k^{(i)})$  as well as compute an error bound for the estimate. The error bound and gradient estimation method apply to any function which satisfies Assumption 2. This method is a significant improvement of our previous work (Michael et al., 2020) and generalises to any dimension with any number of neighbours. Furthermore, we emphasise that the results from Section 3 are independent of this section. The results presented here are an example of one possible method of gradient estimation and estimation error bounding. We make the following assumption on the neighbour set.

**Assumption 13.** For each agent  $i \in \mathcal{V}$  with state  $\mathbf{x}_k^{(i)} \in \mathbb{R}^d$ , the neighbour set cardinality satisfies  $|\mathcal{N}^{(i)}| \geq d$ . Further, the vectors  $\{\mathbf{x}_k^{(j)} - \mathbf{x}_k^{(i)}\}_{j \in \mathcal{N}^{(i)}}$  span  $\mathbb{R}^d$ .

The requirement that the agents do not arrange on a low dimensional subspace is one of the primary motivators for incorporating formation control, as well as preventing collisions in applications with physical robots. Similar requirements for the arranging of agents, and controllers to achieve non-collinearity, are discussed in Bishop, Anderson, Fidan, Pathirana, and Mao (2009), Liu and de Queiroz (2021), Ogren et al. (2004) and Shames, Bishop, Smith, and Anderson (2013).

**Remark 2.** In the absence of Assumption 13, it is still possible to compute an approximate gradient using a variety of methods, such as in (9). However, it is not possible to bound the error of the gradient estimate.

We define three important variables before proceeding:

$$\begin{aligned} \mathbf{s}_k^{(ij)} &:= \frac{f_k(\mathbf{x}_k^{(j)}) - f_k(\mathbf{x}_k^{(i)})}{\|\mathbf{x}_k^{(j)} - \mathbf{x}_k^{(i)}\|}, & \mathbf{v}_k^{(ij)} &:= \frac{\mathbf{x}_k^{(j)} - \mathbf{x}_k^{(i)}}{\|\mathbf{x}_k^{(j)} - \mathbf{x}_k^{(i)}\|}, \\ \mathbf{a}_k^{(ij)} &:= \frac{L_f}{2} \|\mathbf{x}_k^{(j)} - \mathbf{x}_k^{(i)}\|. \end{aligned} \quad (6)$$

We use  $\mathbf{s}_k^{(i)}$ ,  $\mathbf{a}_k^{(i)}$  to denote the vertically stacked vectors of  $\mathbf{s}_k^{(ij)}$ ,  $\mathbf{a}_k^{(ij)}$  for all neighbours  $j \in \mathcal{N}^{(i)}$ . The proof of the following lemma is given in Appendix D.

**Lemma 14.** For a function  $f_k$  satisfying Assumption 2 and an agent  $i$  with neighbour set  $\mathcal{N}^{(i)}$  satisfying Assumption 13, there exists a bounded polyhedron  $\mathcal{P}_k^{(i)} := \{x \in \mathbb{R}^d \mid \begin{bmatrix} A_k^{(i)} \\ -A_k^{(i)} \end{bmatrix} x \leq b_k^{(i)}\}$  such that  $\nabla f(\mathbf{x}_k^{(i)}) \in \mathcal{P}_k^{(i)}$ , for  $A_k^{(i)} \in \mathbb{R}^{|\mathcal{N}^{(i)}| \times d}$  and  $b_k^{(i)} \in \mathbb{R}^{2|\mathcal{N}^{(i)}| \times d}$ .

From Lemma 14, there exists a bounded  $\mathcal{P}_k^{(i)}$  within which the gradient  $\nabla f(\mathbf{x}_k^{(i)})$  must exist. In Michael et al. (2020), we restricted the error bound analysis to 2 dimensions with 2 neighbours. The same method is not computationally feasible in higher dimension, as it requires computation of the largest diagonal in the  $d$ -parallelotope, which has  $2^{d-1}$  diagonals. Instead, we define the following ellipse

$$m_k^{(i)} := \sqrt{\sum_{j \in \mathcal{N}^{(i)}} (|\mathbf{s}_k^{(ij)} - (\mathbf{g}_k^{(i)})^T \mathbf{v}_k^{(ij)}| + d_k^{(ij)})^2} \quad (7)$$

$$\mathcal{E}_k^{(i)} := \left\{ x \in \mathbb{R}^d \mid \left\| \frac{A_k^{(i)}(x - \mathbf{g}_k^{(i)})}{m_k^{(i)}} \right\|^2 \leq 1 \right\}, \quad (8)$$

with  $\mathbf{g}_k^{(i)}$  the centre of  $\mathcal{E}_k^{(i)}$ ,  $A_k^{(i)}$  the matrix defined in Lemma 14, and  $\mathbf{s}_k^{(ij)}$ ,  $\mathbf{v}_k^{(ij)}$ ,  $d_k^{(ij)}$  defined in (6). Define

$$\mathbf{g}_k^{(i)} := ((A_k^{(i)})^T A_k^{(i)})^\dagger (A_k^{(i)})^T \mathbf{s}_k^{(i)}. \quad (9)$$

Note that the centre of the ellipse will serve as the gradient estimate for agent  $i$ , and is equivalent to the simplex gradient (Regis, 2015) of agent  $i$  and its neighbours. We use the superscript  $\dagger$  to denote the Moore Penrose pseudo-inverse, which is equivalent to the inverse when Assumption 13 holds. In the following theorem (proof in Appendix E) we present an error bound that is valid in arbitrary dimension for any number of neighbours.

**Theorem 15.** For a function  $f_k$  satisfying Assumption 2 and an agent  $i$  with neighbour set  $\mathcal{N}^{(i)}$  satisfying Assumption 13, let  $\mathcal{P}_k^{(i)}$  be the polytope defined in Lemma 14. Then  $\mathcal{P}_k^{(i)} \subseteq \mathcal{E}_k^{(i)}$ , for  $\mathcal{E}_k^{(i)}$  the ellipse defined in (8) with centre  $\mathbf{g}_k^{(i)}$  defined in (9). Further, if  $|\mathcal{N}^{(i)}| = d$ ,

and we assume  $B(r, c) = \{x \in \mathbb{R}^d \mid \|x - c\|_2 \leq r\}$  is the smallest bounding ball such that  $\mathcal{P}_k^{(i)} \subseteq B(r, c)$ , then  $\frac{\|a_k^{(i)}\|}{\sigma_{\max}(A_k^{(i)})} \leq r \leq \frac{\|a_k^{(i)}\|}{\sigma_{\min}(A_k^{(i)})}$ , for  $\sigma_{\max/\min}$  the largest/smallest singular values of  $A_k^{(i)}$  and  $a_k^{(i)}$  the vector of  $a_k^{(ij)}$  for all  $j \in \mathcal{N}^{(i)}$ .

The result in [Theorem 15](#) may be interpreted as “the radius of the smallest bounding ball lies between the lengths of semi-major and semi-minor axes of  $\mathcal{E}_k^{(i)}$ ”.

Given that finding the smallest bounding ball which contains a polytope is an NP hard problem, even for the relatively simple centrally symmetric parallelotopes ([Bodlaender, Gritzmman, Klee, & Van Leeuwen, 1990](#)), this approximation is sufficient for the primary goal of gradient estimation. Further, this approximation method gives the smallest 2-norm bound on the error in the simplest case, with  $d$  neighbours distributed in a lattice around agent  $i$ , as demonstrated in [Corollary 16](#).

**Corollary 16.** *If agent  $i$  has neighbour set with cardinality  $|\mathcal{N}^{(i)}| = d$ , and  $(v_k^{(ij)})^T v_k^{(il)} = 0$  for all  $j, l \in \mathcal{N}^{(i)}$  with  $j \neq l$ , then  $\mathcal{E}_k^{(i)}$  as defined in [\(8\)](#) is the smallest bounding ball such that  $\mathcal{P}_k^{(i)} \in \mathcal{E}_k^{(i)}$ .*

**Proof.** If all neighbours are orthogonal, then  $A_k^{(i)}$  as defined in [Lemma 14](#) is an orthogonal matrix, i.e.  $(A_k^{(i)})^T A_k^{(i)} = I$ . Therefore,  $\mathcal{E}^{(i)}$  is a ball. Further, from [Theorem 15](#), the smallest bounding ball radius lies between the largest and smallest radii of  $\mathcal{E}_k^{(i)}$ , which in this case are the same radius. Therefore,  $\mathcal{E}^{(i)}$  is the smallest bounding ball containing  $\mathcal{P}_k^{(i)}$ .

For any number of neighbours satisfying [Assumption 13](#), [Theorem 15](#) guarantees a gradient estimation error bound of the form  $\|g_k^{(i)} - \nabla f_k(x_k^{(i)})\| \leq \frac{m_k^{(i)}}{\sigma_{\min}(A_k^{(i)})}$ , for  $g_k^{(i)}$  the estimated gradient [\(9\)](#) and  $m_k^{(i)}$  as defined in [\(7\)](#). Note that if [Assumption 13](#) does not hold, then  $A_k^{(i)}$  is a low rank matrix, with a minimal singular value of 0, and thus the bound is undefined.

#### 4.1. Bounding ellipse for large neighbour sets

The ellipse from [\(8\)](#) performs well for smaller sets of neighbours, but tends to be conservative when the neighbour set is larger than  $d$ . We provide an additional bounding ellipse here, which shares many of the useful properties of the ellipse defined in [\(8\)](#), but tends to be significantly less conservative in larger problems. The potentially large scaling factor in the denominator of [\(8\)](#) is distributed to each row, rather than applied uniformly, which mitigates some of the inflation from redundant neighbours. We define a matrix  $B_k^{(i)} \in \mathbb{R}^{|\mathcal{N}^{(i)}| \times d}$  with the  $j$ th row  $B_k^{(i)}[j]$  defined  $B_k^{(i)}[j] := \frac{(v_k^{(ij)})^T}{\sqrt{|\mathcal{N}^{(i)}|(|s_k^{(ij)} - (g_k^{(i)})^T v_k^{(ij)}| + d_k^{(ij)})}}$  for  $g_k^{(i)} \in \mathbb{R}^d$  the centre of the ellipse. The second ellipsoidal approximation of  $\mathcal{P}_k^{(i)}$  can then be defined as

$$\tilde{\mathcal{E}}_k^{(i)} := \left\{ x \in \mathbb{R}^d \mid \left\| B_k^{(i)}(x - g_k^{(i)}) \right\|^2 \leq 1 \right\}. \quad (10)$$

It can be verified that  $\tilde{\mathcal{E}}_k^{(i)}$  defined in [\(10\)](#) also contains  $\mathcal{P}_k^{(i)}$ . However, the radius of the smallest bounding ball is not guaranteed to lie between the largest and smallest eigenvalues, and thus  $\tilde{\mathcal{E}}_k^{(i)}$  does not satisfy the claims of [Corollary 16](#). For problems with larger sets of neighbours, the authors have empirically observed that  $\tilde{\mathcal{E}}_k^{(i)}$  seems to be a tighter approximation of  $\mathcal{P}_k^{(i)}$ .

## 5. Simulations

In this section we provide numerical studies to illustrate the results from the previous sections, as well as comparison to another distributed extremum seeking algorithm. For the time varying scalar field, we use convex quadratic functions  $f_k(x) = \frac{1}{2}(x - c(k))^T Q(x - c(k)) + \zeta^T(x - c(k)) + p$ , for positive semi-definite  $Q$ . The values used in the following plots are  $Q = \begin{bmatrix} 2.66 & -0.36 \\ -0.35 & 1.74 \end{bmatrix}$ ,  $\zeta = [-1.28, 4.66]^T$ ,  $p = 6.26$ ,  $c(k) = 10 \sin(\frac{\sqrt{2}k}{100}) + 10 \sin(\frac{\sqrt{3}k}{100}) + \frac{k}{100}$ , with  $L_f, \mu_f$  the largest and smallest eigenvalues of  $Q$  respectively. For the formation control function, we designate a set of neighbours for each agent  $\mathcal{N}^{(i)}$  along with a corresponding set of ideal displacements  $\hat{x}^{(ij)}$ . The formation potential function is then

$$\phi(\mathbf{x}_k) = \phi^* + L_f \sum_{i \in \mathcal{V}} \sum_{j \in \mathcal{N}^{(i)}} \|x^{(i)} - x^{(j)} - \hat{x}^{(ij)}\|_2^2. \quad (11)$$

For other potential functions which satisfy the definitions used here, see [De Gennaro and Jadbabaie \(2006\)](#), [Dimarogonas et al. \(2006\)](#), [Do \(2006\)](#), [Olfati-Saber and Murray \(2002\)](#) and [Tanner and Kumar \(2005\)](#). In [Michael et al. \(2020\)](#) we derive the error bound on the gradient estimation in two dimensions, and show that the estimation error is proportional to the distance between the agents, with proportionality constant  $L_f$ , so the Lipschitz constant  $L_f$  and the minimum value  $\phi^*$  in [\(11\)](#) ensure that  $\phi(\mathbf{x}_k)$  satisfies [Assumption 10](#). The minimum value  $\phi^*$  is chosen as an upper bound on the gradient approximation error when the agents are in perfect formation, derived from the gradient estimation error bounds in [Theorem 15](#).

The simulated methods include the composite method derived in [Section 3](#) using two different formations, as well as the consensus for circular formations from [Brinón-Arranz et al. \(2015\)](#) for comparison. For the composite method, as described in [Section 3](#), we use the simplex gradient as the local gradient estimation method at each iteration ([Algorithm 1](#)).

---

### Algorithm 1 Distributed Composite Dynamics

---

```

for  $k = 1, 2, \dots$  do
  for  $i \in \{1, 2, \dots, n\}$  do
     $g_k^{(i)} = ((A_k^{(i)})^T A_k^{(i)})^\dagger (A_k^{(i)})^T \mathbf{s}_k^{(i)}$ 
  for  $i \in \{1, 2, \dots, n\}$  do
     $x_{k+1}^{(i)} = x_k^{(i)} - \frac{1}{L} (g_k^{(i)} + \nabla_{x_k^{(i)}} \phi(\mathbf{x}_k))$ 

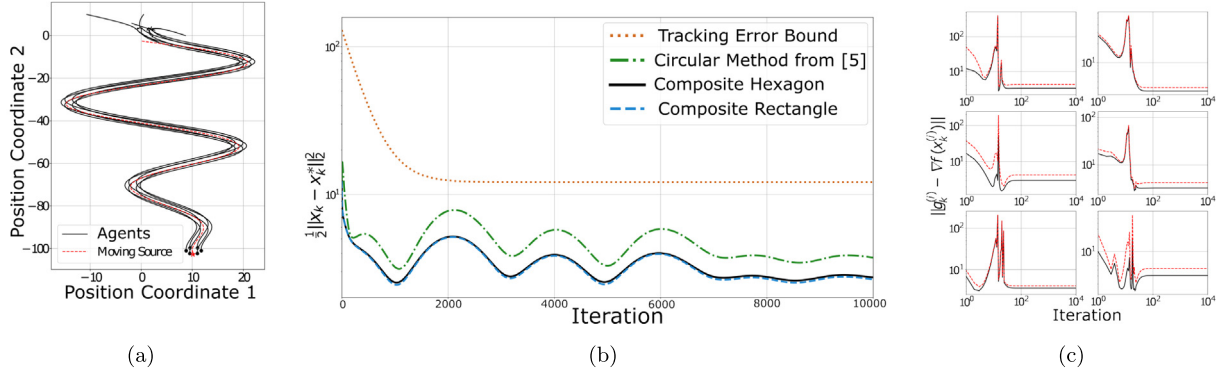
```

---

The circular formation controller is presented in [Algorithm 2](#), and is written exactly as in [Brinón-Arranz et al. \(2015\)](#) accounting for the notation of this paper. The parameters used within [Algorithm 2](#) are the same as used in the original paper ([Brinón-Arranz et al., 2015](#)), in the example provided therein without noise. The radius of the formation  $D = 3$ , the rotation velocity  $\omega = 1$ ,  $\epsilon = 0.5$  and  $\alpha = 1$ . The consensus matrix used is also the same as [Brinón-Arranz et al. \(2015\)](#), with weight equal to 0.25.

Choosing six agents forces the use of a regular hexagon for [Brinón-Arranz et al. \(2015\)](#). We therefore included the composite method using a regular hexagon formation for comparison.

While the circular motion controller in [Brinón-Arranz et al. \(2015\)](#) requires this hexagonal arrangement for six agents, the framework proposed in this paper is flexible in the choice of formation by changing the ideal displacements  $\hat{x}^{(ij)}$ . To this end we also include a rectangular formation obtained from the tessellation of two squares along a common edge. As shown in [Michael et al. \(2020\)](#), the gradient estimation error bound is a function of the orthogonality of the neighbours as well as the distance



**Fig. 1.** (a) Agent trajectories using the composite method from Section 3. (b) Formation distance from the signal source. (c) Gradient estimation error (black) and estimation error bound of Theorem 15 (red) for each agent from (a). (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

### Algorithm 2 Circular Source Seeking

```

for  $i = 1, \dots, n$  do
   $h_0^{(i)} = \tilde{g}_0^{(i)} = h_{-1}^{(i)} = c_0^{(i)} + f_0(x_0^{(i)})(x_0^{(i)} - c_0^{(i)})$ 
   $\phi^{(i)} = i \frac{2\pi}{n}$ 
for  $k = 1, 2, \dots$  do
  for  $i = 1, \dots, n$  do
     $g_k^{(i)} = c_k^{(i)} + \frac{2}{D^2} f(x_k^{(i)})(x_k^{(i)} - c_k^{(i)})$ 
     $\tilde{g}_k^{(i)} = (1 - \alpha) \tilde{g}_{k-1}^{(i)} + \alpha g_k^{(i)}$ 
     $\tilde{h}_k^{(i)} = h_{k-1}^{(i)} + \tilde{g}_{k-1}^{(i)} - \tilde{g}_{k-2}^{(i)}$ 
   $h_k = (P \otimes I_2)(\tilde{h}_k)$ 
for  $i = 1, \dots, n$  do
     $c_k^{(i)} = (1 - \varepsilon) c_{k-1}^{(i)} + \varepsilon h_k^{(i)}$ 
     $x_k^{(i)} = c_k^{(i)} + DR(\phi^{(i)} + \omega k)$ 

```

between them, so the rectangular formation will have lower gradient estimation error than the hexagonal formation with the same neighbour distances. Fig. 1a shows the resulting trajectories from the composite method. We exclude the trajectories from other methods, as they are visually identical. Instead, we include the comparison of the tracking error  $\frac{1}{2}d(x_{k+1}, x_{k+1}^*)^2$  in Fig. 1b for each method, including the theoretical bounds from Theorem 12. The circular formation has higher tracking error and the rectangular and hexagonal formations using the composite method track nearly identically, although the rectangular formation converges slightly closer to the optimal value (Fig. 1b) set due to the lower gradient error as illustrated in Fig. 1c.

As the results from Section 3 generalise to any dimension, we provide an example in three dimensions, as well as an implementation of the extremum seeking algorithm from Section 3, at the provided link.<sup>1</sup>

## 6. Conclusion

In this paper we consider a formation of agents tracking the optimum of a time varying scalar field with no gradient information, in arbitrary dimension. At each iteration, the agents take measurements, communicate with their neighbours to estimate a descent direction, and converge to a neighbourhood of the optimum. We derive distributed control laws which drive the agents to a bounded neighbourhood of the optimiser set, without the delineation of leaders/followers or the use of communication intensive consensus protocols. The method is flexible to the

choice of formation and gradient estimation method, and we provide examples using two formations and gradient estimation using the simplex gradient. By blending formation control with extremum seeking, the agents are able to minimise the gradient estimation error, improving the neighbourhood of convergence. We concluded with numerical studies showing that the proposed method is comparable with other extremum seeking methods, converging to a tighter neighbourhood while being more flexible in the choice of formation. Further research will focus on the relaxing of the assumptions on the formation potential functions, allowing for potential functions with non unique minima which do not satisfy the PL inequality, and incorporating time-varying neighbour sets.

## Appendix A. Proof of Lemma 8

The agent identifying superscript  $i$  is suppressed in this proof, as all calculations correspond to a single agent. Assumption 2 along with (2) results in  $f_k(x_{k+1}) - f_k(x_k) \leq \nabla f_k(x_k)^T(x_{k+1} - x_k) + \frac{L_f}{2} \|x_{k+1} - x_k\|^2$ . In turn,  $\nabla f_k(x_k)^T(x_{k+1} - x_k) + \frac{L_f}{2} \|x_{k+1} - x_k\|^2 = -\alpha \nabla f_k(x_k)^T(\nabla f_k(x_k) + \varepsilon_k) + \frac{\alpha^2 L_f}{2} \|\nabla f_k(x_k) + \varepsilon_k\|^2$ . Adding and subtracting  $\frac{\alpha}{2} \|\varepsilon_k\|^2$  to complete the square leads to  $f_k(x_{k+1}) - f_k(x_k) \leq \frac{\alpha}{2} \|\varepsilon_k\|^2 - \frac{\alpha}{2} \|\nabla f_k(x_k)\|^2 + \frac{\alpha}{2} (\alpha L_f - 1) \|\nabla f_k(x_k) + \varepsilon_k\|^2$ . From  $\alpha \in (0, \frac{1}{L_f}]$ ,  $\alpha L_f - 1 \leq 0$ ,  $f_k(x_{k+1}) - f_k(x_k) \leq -\frac{\alpha}{2} \|\nabla f_k(x_k)\|^2 + \frac{\alpha}{2} \|\varepsilon_k\|^2$ , and from the Polyak–Łojasiewicz bounds (Assumption 3), one obtains  $f_k(x_{k+1}) - f_k(x_k) \leq -\alpha \mu_f (f_k(x_k) - f_k^*) + \frac{\alpha}{2} \|\varepsilon_k\|^2$ . Adding  $f_{k+1}(x_{k+1}) - f_k(x_{k+1}) + f_k(x_k) - f_{k+1}^*$  to both sides, and using the scalar bounds from Assumption 4:  $f_{k+1}(x_{k+1}) - f_{k+1}^* \leq f_k(x_k) - f_{k+1}^* + f_{k+1}(x_{k+1}) - f_k(x_{k+1}) - \alpha \mu_f (f_k(x_k) - f_k^*) + \frac{\alpha}{2} \|\varepsilon_k\|^2$ . Thus,  $f_{k+1}(x_{k+1}) - f_{k+1}^* \leq f_k(x_k) - f_k^* - \alpha \mu_f (f_k(x_k) - f_k^*) + \frac{\alpha}{2} \|\varepsilon_k\|^2 + \eta^* + \eta_0$ . Consequently,

$$f_{k+1}(x_{k+1}) - f_{k+1}^* \leq (1 - \alpha \mu_f)(f_k(x_k) - f_k^*) + \frac{\alpha}{2} \|\varepsilon_k\|^2 + \eta^* + \eta_0, \quad (A.1)$$

$$f_{k+1}(x_{k+1}) - f_{k+1}^* \leq (1 - \alpha \mu_f)^k (f_0(x_0) - f_0^*) + \frac{\alpha}{2} \sum_{t=0}^k (1 - \alpha \mu_f)^{k-t} \|\varepsilon_t\|^2 + (\eta^* + \eta_0) \frac{1 - (1 - \alpha \mu_f)^k}{\mu_f \alpha}. \quad (A.2)$$

Converting this sub-optimality bound into a bound on the convergence neighbourhood, we use the relationships (4) to obtain the final result.

<sup>1</sup> <https://tinyurl.com/yc4fzpv2>.

## Appendix B. Proof of Lemma 11

Let  $x_\phi^* \in \mathcal{X}_\phi^*$  and  $x_{F_k}^* \in \mathcal{X}_{F_k}^*$  be any of the points satisfying  $\|x_\phi^* - x_{F_k}^*\| = d(\mathcal{X}_\phi^*, \mathcal{X}_{F_k}^*)$ . The Lipschitz properties of  $F_k$  and  $\phi$  lead to  $F_k(x_\phi^*) \leq \frac{L_f}{2} \|x_{F_k}^* - x_\phi^*\|^2 = \frac{L_f}{2} d(\mathcal{X}_\phi^*, \mathcal{X}_{F_k}^*)^2$  and  $\phi(x_{F_k}^*) \leq \phi^* + \frac{L_\phi}{2} \|x_{F_k}^* - x_\phi^*\|^2 = \phi^* + \frac{L_\phi}{2} d(\mathcal{X}_\phi^*, \mathcal{X}_{F_k}^*)^2$ . Using these inequalities, we may bound the values of the composite function  $\hat{f}_k$  at both  $x_\phi^*$  and  $x_{F_k}^*$ . Therefore, given that the minimiser satisfies  $\hat{f}_k^* \leq \hat{f}_k(x_{F_k}^*)$  and  $\hat{f}_k^* \leq \hat{f}_k(x_\phi^*)$ , we have  $\hat{f}_k^* \leq \phi^* + \frac{\min(L_f, L_\phi)}{2} d(\mathcal{X}_\phi^*, \mathcal{X}_{F_k}^*)^2$ .

## Appendix C. Proof of Theorem 12

Note that, just as in the proof of Lemma 8, the agent identifying subscript is suppressed for readability as all calculations are with respect to one agent. As  $\hat{f}_k$  shares all of the properties of  $f_k$ , we pick up from (A.1). Substituting the formation potential function  $\frac{1}{c}\phi(\mathbf{x}_k)$  for the error term  $\frac{1}{2}\|\varepsilon_k\|^2$ , yields  $\hat{f}_{k+1}(\mathbf{x}_{k+1}) - \hat{f}_{k+1}^* \leq (1 - \alpha\mu_f)(\hat{f}_k(\mathbf{x}_k) - \hat{f}_k^*) + \frac{\alpha}{c}\phi(\mathbf{x}_k) + \eta^* + \eta_0$ . Adding the strictly positive term  $\frac{\alpha}{c}(f_k(\mathbf{x}_k) - \hat{f}_k^* + \hat{f}_k^*)$  to the right side of the inequality results in  $\hat{f}_{k+1}(\mathbf{x}_{k+1}) - \hat{f}_{k+1}^* \leq (1 - \alpha(\mu_f - \frac{1}{c}))(\hat{f}_k(\mathbf{x}_k) - \hat{f}_k^*) + \frac{\alpha}{c}\hat{f}_k^* + \eta^* + \eta_0$ . Expanding the recursive relationship, with  $\mu' := \mu_f - \frac{1}{c} \geq 0$ , in terms of the initial conditions leads to

$$\begin{aligned} \hat{f}_{k+1}(\mathbf{x}_{k+1}) - \hat{f}_{k+1}^* &\leq (1 - \alpha\mu')^k (\hat{f}_0(\mathbf{x}_0) - \hat{f}_0^* - \eta^* - \eta_0) \\ &\quad + \frac{\alpha}{c} \sum_{t=0}^k (1 - \alpha\mu')^{k-t} \hat{f}_t^* + \frac{\eta^* + \eta_0}{\alpha\mu'}. \end{aligned} \quad (\text{C.1})$$

Using (4), as in the proof of Lemma 8, we have  $\frac{1}{2}d(x_{k+1}, \mathcal{X}_{F_{k+1}}^*)^2 \leq \frac{(1 - \alpha\mu')^k}{\mu_f} (\frac{L_f}{2} d(x_0, \mathcal{X}_{F_0}^*)^2 - \eta^* - \eta_0) + \frac{\alpha}{c\mu_f} \sum_{t=0}^k (1 - \alpha\mu')^{k-t} \hat{f}_t^* + \frac{\eta^* + \eta_0}{\mu_f \mu' \alpha}$ . Taking the limit completes the proof.

## Appendix D. Proof of Lemma 14

We begin by constructing the polyhedron  $\mathcal{P}_k^{(i)}$ , and showing that  $\nabla f_k(x_k^{(i)}) \in \mathcal{P}_k^{(i)}$ . We then show that, if Assumption 13 holds, the polyhedron is bounded. None of the following analysis spans iterations, so we suppress the iteration counter  $k$  for simplicity.

Consider agents  $x^{(i)}, x^{(j)} \in \mathbb{R}^d$  with  $j \in \mathcal{N}^{(i)}$ . By the mean value theorem,  $\exists t \in [0, 1]$  such that  $\nabla f((1-t)x^{(i)} + tx^{(j)})^T v^{(ij)} = \frac{f(x^{(j)}) - f(x^{(i)})}{\|x^{(j)} - x^{(i)}\|}$ , for  $v^{(ij)}$  defined in (6). The right-hand-side of this equality, which is the same as  $s^{(ij)}$ , is used to estimate the true directional derivative at  $x^{(i)}$ . Combining it with Assumption 2 gives a bound on the error of the directional derivative estimation,

$$\|\nabla f(x^{(i)})^T v^{(ij)} - s^{(ij)}\| \leq \frac{L_f}{2} \|x^{(ij)}\| = a^{(ij)}. \quad (\text{D.1})$$

We may rearrange (D.1) into a pair of inequalities

$$\begin{aligned} (v^{(ij)})^T \nabla f(x^{(i)}) &\leq s^{(ij)} + a^{(ij)} \\ (-v^{(ij)})^T \nabla f(x^{(i)}) &\leq a^{(ij)} - s^{(ij)}. \end{aligned} \quad (\text{D.2})$$

The two inequalities in (D.2) represent two hyperplanes within which the gradient is constrained. The two are oriented by the normal vector  $v^{(ij)}$ , separated by  $2a^{(ij)}$ , and centred on the plane  $(v^{(ij)})^T x = s^{(ij)}$ . Define the matrix  $A \in \mathbb{R}^{|\mathcal{N}^{(i)}| \times d}$ , with each row equal to  $v^{(ij)}$  for a neighbour  $j \in \mathcal{N}^{(i)}$ , and a vector  $b \in \mathbb{R}^{2|\mathcal{N}^{(i)}|}$ , with  $s^{(ij)} + a^{(ij)}$  for each neighbour  $j \in \mathcal{N}^{(i)}$  stacked above  $a^{(ij)} - s^{(ij)}$  for each neighbour. Then the definition of the polyhedron  $\mathcal{P}^{(i)}$  from Lemma 14 represents the set of  $2|\mathcal{N}^{(i)}|$  inequalities from (D.2), and  $\nabla f(x^{(i)}) \in \mathcal{P}^{(i)}$ .

To see that the polyhedron is bounded, let  $\{e_1, e_2, \dots, e_n\}$  be the set of canonical basis vectors in  $\mathbb{R}^d$ . By Assumption 13, the

vectors  $\{v^{(ij)}\}_{j \in \mathcal{N}^{(i)}}$  span  $\mathbb{R}^d$ , and we may express each basis vector by a linear combination  $e_l = \sum_{j \in \mathcal{N}^{(i)}} c_l^{(j)} v^{(ij)}$ . We then have, for each point  $x \in \mathcal{P}$ ,  $e_l^T x = \sum_{j \in \mathcal{N}^{(i)}} (c_l^{(j)} v^{(ij)})^T x \leq \sum_{j \in \mathcal{N}^{(i)}} c_l^{(j)} (s^{(ij)} + a^{(ij)})$ . Using the first inequality from (D.2). In the negative  $e_l$  direction we make use of the second inequality in (D.2). Hence,  $-e_l^T x = \sum_{j \in \mathcal{N}^{(i)}} (c_l^{(j)} (-v^{(ij)}))^T x \leq \sum_{j \in \mathcal{N}^{(i)}} c_l^{(j)} (a^{(ij)} - s^{(ij)})$ . We therefore have that, if Assumption 13 holds, the polyhedron is bounded in  $\mathbb{R}^d$ .

## Appendix E. Proof of Theorem 15

Once again, we suppress the iteration identifying subscript  $k$ , as all the analysis takes place in a single iteration. Define a shifted coordinate system  $y = x - g^{(i)}$ , with the centre of the ellipse  $g^{(i)}$  as the origin. The inequalities defining the interior of the polytope  $\mathcal{P}^{(i)}$  from (D.2) then become

$$\begin{aligned} (v^{(ij)})^T y &\leq a^{(ij)} + s^{(ij)} - (g^{(i)})^T v^{(ij)} \\ (-v^{(ij)})^T y &\leq a^{(ij)} - (s^{(ij)} - (g^{(i)})^T v^{(ij)}). \end{aligned} \quad (\text{E.1})$$

Let  $y \in \mathcal{P}^{(i)}$  be any point within the polytope, i.e. it satisfies (E.1) for all  $j \in \mathcal{N}^{(i)}$ . Then either  $(y^T v^{(ij)})^2 \leq (s^{(ij)} - (g^{(i)})^T v^{(ij)} + a^{(ij)})^2$ , or  $(y^T v^{(ij)})^2 \leq (a^{(ij)} - (s^{(ij)} - (g^{(i)})^T v^{(ij)}))^2$ , depending on the sign of  $s^{(ij)} - (g^{(i)})^T v^{(ij)}$ . We may then use the single inequality

$$(y^T v^{(ij)})^2 \leq (|s^{(ij)} - (g^{(i)})^T v^{(ij)}| + a^{(ij)})^2, \quad (\text{E.2})$$

for any point  $y \in \mathcal{P}^{(i)}$ . Given the matrix  $A^{(i)}$  as defined in Lemma 14, we have  $y^T (A^{(i)})^T A^{(i)} y = \sum_{j \in \mathcal{N}^{(i)}} y^T v^{(ij)} (v^{(ij)})^T y = \sum_{j \in \mathcal{N}^{(i)}} ((v^{(ij)})^T y)^2$ . Assuming  $y \in \mathcal{P}^{(i)}$  and applying (E.2), one obtains  $y^T (A^{(i)})^T A^{(i)} y \leq \sum_{j \in \mathcal{N}^{(i)}} (|s^{(ij)} - (g^{(i)})^T v^{(ij)}| + a^{(ij)})^2$ . Consequently,  $\|A^{(i)} y\|^2 \leq (m_k^{(i)})^2$  for  $m_k^{(i)}$  defined in (7). Therefore, each point in the polytope  $\mathcal{P}^{(i)}$  is in the ellipse (8). Note that this works for any centre  $g^{(i)}$ , but the resulting ellipse will be differently sized depending on the choice of  $g^{(i)}$ .

We now assume that  $|\mathcal{N}^{(i)}| = d$ , and therefore  $\mathcal{P}^{(i)}$  is a  $d$ -parallelotope, with parallel and congruent opposite faces. The centre of the parallelotope  $c$  is the point  $A^{(i)} c = s^{(i)}$ , for  $s^{(i)}$  the vector of  $s^{(ij)}$  for all  $j \in \mathcal{N}^{(i)}$ . This is the point at which all diagonals intersect, and are bisected, and thus must be the centre of the smallest bounding ball  $\mathcal{B}^{(i)}(r, c)$ . We note this point is also returned by (9), therefore the ellipse  $\mathcal{E}^{(i)}$  and the smallest bounding ball share the same centre. We may then assume, without loss of generality, that the parallelotope is centred at the origin. This further simplifies the definition of  $\mathcal{E}^{(i)}$ , as the term  $\sum_{j \in \mathcal{N}^{(i)}} (|s_k^{(ij)} - (g^{(i)})^T v_k^{(ij)}| + a_k^{(ij)})^2 = \|a^{(i)}\|^2$  for  $a^{(i)}$  the vector of  $a^{(ij)}$  for all  $j \in \mathcal{N}^{(i)}$  as defined in (6). Let  $V_k \in \mathbb{R}^{n \times n}$  be a diagonal matrix with  $V_{ii} \in \{-1, 1\}$ . Then the vertices of  $\mathcal{P}^{(i)}$  are the points  $v_k = (A^{(i)})^{-1} V_k a^{(i)}$ ,  $\forall k \in \{1, 2, 3, \dots, 2^n\}$ .

The smallest bounding ball, by definition, includes all these vertices and therefore  $r \geq \max_{v_k} \|(A^{(i)})^{-1} V_k a^{(i)}\| \geq \sigma_{\min}((A^{(i)})^{-1}) \|V_k a^{(i)}\| = \frac{\|a^{(i)}\|}{\sigma_{\max}(A^{(i)})}$ . Furthermore,  $\mathcal{B}(c, r)$  radius is smaller than the length of the semi-major axis of  $\mathcal{E}^{(i)}$ . The length of the semi-major axis  $\mathcal{E}^{(i)}$  corresponds to the inverse of the smallest singular value of the shape matrix, i.e.  $r \leq \frac{\|a^{(i)}\|}{\sigma_{\min}(A^{(i)})}$ . Combining these results completes the proof.

## References

- Al Issa, H. A. (2012). *Position-adaptive direction finding for multi-platform RF emitter localization using extremum seeking control* (Ph.D. thesis), University of Dayton.
- Ariyur, K. B., & Krstic, M. (2003). *Real-time optimization by extremum-seeking control*. John Wiley & Sons.
- Ben-Israel, A., & Mond, B. (1986). What is invexity? *The ANZIAM Journal*, 28(1), 1-9.

- Bishop, A. N., Anderson, B. D., Fidan, B., Pathirana, P. N., & Mao, G. (2009). Bearing-only localization using geometrically constrained optimization. *IEEE Transactions on Aerospace and Electronic Systems*, 45(1), 308–320.
- Biyük, E., & Arcaç, M. (2008). Gradient climbing in formation via extremum seeking and passivity-based coordination rules. *Asian Journal of Control*, 10(2), 201–211.
- Bodlaender, H. L., Gritzmann, P., Klee, V., & Van Leeuwen, J. (1990). Computational complexity of norm-maximization. *Combinatorica*, 10(2), 203–225.
- Brinón-Arranz, L., & Schenato, L. (2013). Consensus-based source-seeking with a circular formation of agents. In *2013 European control conference ECC*, (pp. 2831–2836). IEEE.
- Brinón-Arranz, L., Schenato, L., & Seuret, A. (2015). Distributed source seeking via a circular formation of agents under communication constraints. *IEEE Transactions on Control of Network Systems*, 3(2), 104–115.
- Brinón-Arranz, L., Seuret, A., & Canudas-de Wit, C. (2011). Collaborative estimation of gradient direction by a formation of AUVs under communication constraints. In *2011 50th IEEE conference on decision and control and European control conference* (pp. 5583–5588). IEEE.
- Cochran, J., & Krstic, M. (2009). Nonholonomic source seeking with tuning of angular velocity. *IEEE Transactions on Automatic Control*, 54(4), 717–731.
- De Gennaro, M. C., & Jadbabaie, A. (2006). Decentralized control of connectivity for multi-agent systems. In *Proceedings of the 45th IEEE conference on decision and control* (pp. 3628–3633). IEEE.
- Dimarogonas, D. V., Loizou, S. G., Kyriakopoulos, K. J., & Zavlanos, M. M. (2006). A feedback stabilization and collision avoidance scheme for multiple independent non-point agents. *Automatica*, 42(2), 229–243.
- Do, K. (2006). Formation stabilization and tracking control of mobile agents using local potential functions. In *2006 American control conference* (pp. 6–pp). IEEE.
- Dürr, H.-B., Krstić, M., Scheinker, A., & Ebenbauer, C. (2017). Extremum seeking for dynamic maps using Lie brackets and singular perturbations. *Automatica*, 83, 91–99.
- Dürr, H.-B., Stanković, M. S., Ebenbauer, C., & Johansson, K. H. (2013). Lie bracket approximation of extremum seeking systems. *Automatica*, 49(6), 1538–1552.
- Frihauf, P., Liu, S.-J., & Krstic, M. (2014). A single forward-velocity control signal for stochastic source seeking with multiple nonholonomic vehicles. *Journal of Dynamic Systems, Measurement, and Control*, 136(5), Article 051024.
- Ghods, N., & Krstic, M. (2011). Multiagent deployment over a source. *IEEE Transactions on Control Systems Technology*, 20(1), 277–285.
- Grushkovskaya, V., Dürr, H.-B., Ebenbauer, C., & Zuyev, A. (2017). Extremum seeking for time-varying functions using Lie bracket approximations. *IFAC-PapersOnLine*, 50(1), 5522–5528.
- Hazeleger, L., Haring, M., & van de Wouw, N. (2020). Extremum-seeking control for optimization of time-varying steady-state responses of nonlinear systems. *Automatica*, 119, Article 109068.
- Karimi, H., Nutini, J., & Schmidt, M. (2016). Linear convergence of gradient and proximal-gradient methods under the polyak-tojasiewicz condition. In *Joint European conference on machine learning and knowledge discovery in databases* (pp. 795–811). Springer.
- Khong, S. Z., Tan, Y., Manzie, C., & Nešić, D. (2014). Multi-agent source seeking via discrete-time extremum seeking control. *Automatica*, 50(9), 2312–2320.
- Killingsworth, N. J., Aceves, S. M., Flowers, D. L., Espinosa-Loza, F., & Krstic, M. (2009). HCCI engine combustion-timing control: Optimizing gains and fuel consumption via extremum seeking. *IEEE Transactions on Control Systems Technology*, 17(6), 1350–1361.
- Liu, T., & de Queiroz, M. (2021). An orthogonal basis approach to formation shape control. *Automatica*, 129, Article 109619.
- Mayhew, C. G., Sanfelice, R. G., & Teel, A. R. (2007). Robust source-seeking hybrid controllers for autonomous vehicles. In *2007 American control conference* (pp. 1185–1190). IEEE.
- Michael, E., Zelazo, D., Wood, T. A., Manzie, C., & Shames, I. (2020). Optimisation with zeroth-order oracles in formation. In *2020 59th IEEE conference on decision and control* (pp. 5354–5359). IEEE.
- Moore, B. J., & Canudas-de Wit, C. (2010). Source seeking via collaborative measurements by a circular formation of agents. In *Proceedings of the 2010 American control conference* (pp. 6417–6422). IEEE.
- Moshksar, E., Dougherty, S., & Guay, M. (2015). Model-based extremum-seeking control for unstable systems with time-varying extremum. In *2015 54th IEEE conference on decision and control CDC*, (pp. 6960–6965). IEEE.
- Ogren, P., Fiorelli, E., & Leonard, N. E. (2004). Cooperative control of mobile sensor networks: Adaptive gradient climbing in a distributed environment. *IEEE Transactions on Automatic Control*, 49(8), 1292–1302.
- Olfati-Saber, R., & Murray, R. M. (2002). Distributed cooperative control of multiple vehicle formations using structural potential functions. *IFAC Proceedings Volumes*, 35(1), 495–500.
- Poveda, J. I., & Krstić, M. (2021). Fixed-time seeking and tracking of time-varying extrema. In *2021 60th IEEE conference on decision and control CDC*, (pp. 108–113). IEEE.
- Regis, R. G. (2015). The calculus of simplex gradients. *Optimization Letters*, 9(5), 845–865.
- Shames, I., Bishop, A. N., Smith, M., & Anderson, B. D. (2013). Doppler shift target localization. *IEEE Transactions on Aerospace and Electronic Systems*, 49(1), 266–276.
- Shames, I., Selvaratnam, D., & Manton, J. H. (2019). Online optimization using zeroth order oracles. *IEEE Control Systems Letters*, 4(1), 31–36.
- Skobeleva, A., Fidan, B., Ugrinovskii, V., & Petersen, I. R. (2018). Planar cooperative extremum seeking with guaranteed convergence using a three-robot formation. In *2018 IEEE conference on decision and control CDC*, (pp. 1305–1310). IEEE.
- Stankovic, M. S., Johansson, K. H., & Stipanovic, D. M. (2011). Distributed seeking of Nash equilibria with applications to mobile sensor networks. *IEEE Transactions on Automatic Control*, 57(4), 904–919.
- Tan, Y., Moase, W. H., Manzie, C., Nešić, D., & Mareels, I. M. (2010). Extremum seeking from 1922 to 2010. In *Proceedings of the 29th Chinese control conference* (pp. 14–26). IEEE.
- Tanner, H. G., & Kumar, A. (2005). Formation stabilization of multiple agents using decentralized navigation functions. In *Robotics: Science and systems, Vol. 1* (pp. 49–56). Boston.
- Vandermeulen, I., Guay, M., & McLellan, P. J. (2017). Discrete-time distributed extremum-seeking control over networks with unstable dynamics. *IEEE Transactions on Control of Network Systems*, 5(3), 1182–1192.
- Vweza, A. O., Chong, K. T., & Lee, D. J. (2015). Gradient-free numerical optimization-based extremum seeking control for multiagent systems. *International Journal of Control, Automation and Systems*, 13(4), 877–886.
- Ye, M., & Hu, G. (2016). Distributed extremum seeking for constrained networked optimization and its application to energy consumption control in smart grid. *IEEE Transactions on Control Systems Technology*, 24(6), 2048–2058.
- Zhang, C., Siranosian, A., & Krstić, M. (2007). Extremum seeking for moderately unstable systems and for autonomous vehicle target tracking without position measurements. *Automatica*, 43(10), 1832–1839.
- Zhao, S., & Sun, Z. (2017). Defend the practicality of single-integrator models in multi-robot coordination control. In *2017 13th IEEE international conference on control & automation ICCA*, (pp. 666–671). IEEE.
- Zhao, S., & Zelazo, D. (2015a). Bearing-based formation maneuvering. In *2015 IEEE international symposium on intelligent control ISIC*, (pp. 658–663). IEEE.
- Zhao, S., & Zelazo, D. (2015b). Translational and scaling formation maneuver control via a bearing-based approach. *IEEE Transactions on Control of Network Systems*, 4(3), 429–438.



**Elad Michael** received a B.Sc. degree in Physics and Mathematics in 2016 from Stony Brook University, New York. In 2022 he completed a Ph.D. in Automation and Control Theory at the University of Melbourne. He is currently, and hopefully still, travelling. His research interests are motly, recently focusing on theoretical computer science, strange number sequences, and sugar kelp farming.



**Chris Manzie** is a full Professor and Head of Department of Electrical and Electronic Engineering at the University of Melbourne. Over the period 2003–2016, he was an academic in the Department of Mechanical Engineering, with responsibilities including Assistant Dean with the portfolio of Research Training (2011–2017), and Mechatronics Program Director (2009–2016). Professor Manzie was a Visiting Scholar with the University of California, San Diego in 2007 and a Visiteur Scientifique at IFP Energies Nouvelles, Rueil Malmaison in 2012. His research interests are in model-

based and model-free control and optimisation with applications in a range of areas, including systems related to autonomous systems, energy, transportation, and mechatronics.



**Tony A. Wood** received a B.Sc. degree in Information Technology and Electrical Engineering in 2010 and an M.Sc. degree in Robotics, Systems and Control in 2013, both from ETH Zurich, Switzerland. In 2018 he completed a Ph.D. at the Automatic Control Laboratory of ETH Zurich. From 2018 to 2021, he was a Research Fellow in the Department of Electrical and Electronic Engineering at the University of Melbourne, Australia. He is currently a Scientist in the Systems Control and Multiagent Optimisation Research (SYCAMORE) Lab at École Polytechnique Fédérale de Lausanne (EPFL), Switzerland. His research interests lie in the fields of optimisation and automatic control with a particular focus on multi-agent systems. The specific topics he investigates include system identification, task allocation, path planning, model predictive control, delay compensation, and formal specification satisfaction for



uncertain systems encountered in applications such as energy systems, smart buildings, biology, and robotics.



**Daniel Zelazo** received the B.Sc. and M.Eng. degrees in electrical engineering and computer science from the Massachusetts Institute of Technology, Cambridge, MA, USA, in 1999 and 2001, respectively, and the Ph.D. degree in aeronautics and astronautics from the University of Washington, Seattle, WA, USA, in 2009. He is an Associate Professor of Aerospace Engineering and the Director of the Philadelphia Flight Control Laboratory, Technion—Israel Institute of Technology, Haifa, Israel. From 2010 to 2012, he was a Postdoctoral Research Associate and a Lecturer with the Institute for Systems Theory and Automatic Control, University of Stuttgart, Stuttgart, Germany. His research interests include topics related to multiagent systems. Dr. Zelazo is currently an Associate Editor of IEEE Control Systems Letters and a Subject Editor of the International Journal of Robust and Nonlinear Control.



**Iman Shames** is currently a Professor of Mechatronics and the Mechatronics Cluster lead as well as the CHCADA Lab director at the School of Engineering, the Australian National University. Previously, he had been an Associate Professor at the Department of Electrical and Electronic Engineering, the University of Melbourne from 2014 to 2020 and a Senior Lecturer and a McKenzie fellow at the same department from 2012 to 2014, and before that he was an ACCESS Postdoctoral Researcher at the ACCESS Linnaeus Centre, the KTH Royal Institute of Technology, Stockholm, Sweden. He received his B.Sc. degree in Electrical Engineering from Shiraz University in 2006, and the Ph.D. degree in engineering from the Australian National University, Canberra, Australia in 2011. His current research interests include, but are not limited to, decision making for dynamical systems under uncertainty, optimisation theory and its application in control and estimation, and mathematical systems theory of cyber–physical systems.